GLENN JOY

Squaring the Circles: Lewis Carroll's Diagrams vs. John Venn's Diagrams

"For a *complete* logical argument," Arthur began with admirable solemnity, "we need two prim Misses——"

"Of course!" she interrupted. "I remember that word now. And they produce—?"

"A Delusion," said Arthur.

"Ye—es?" she said dubiously, "I don't seem to remember that so well. But what is the *whole* argument called?"

"A Sillygism."

—Lewis Carroll, *Sylvie and Bruno*

Although Lewis Carroll (The Rev. C. L. Dodgson–Jan. 27, 1832-Jan. 14, 1898) spent his professional academic life as a mathematician at Oxford, he made little lasting contribution to that field. But I think he would be pleased to know that Alice lives today and that his work on logic and puzzles has spawned many discussions. For example, his short "What the Tortoise Said to Achilles" is widely reprinted and still evokes commentary (Carroll 1895). Since Lewis Carroll lived and worked during the era of Boole and DeMorgan (at the beginning of the development of modern logic), his work provides us with a fascinating and exciting way to learn logic and provides a unique window into the world of logic and its applications. His *Symbolic Logic* and *The Game of Logic* are still in print (Carroll, 1958; both books bound as one). In addition, W. W. Bartley, III (1977) pulled together from public and private collections around the world fragments of Carroll's proposed Part II of *Symbolic Logic* into a very nice volume containing both Part I and Part II, although it is no longer in print.

Lewis Carroll's corpus of works is more tightly connected than many persons realize, but clearly his strictly academic works in mathematics are related to his works in logic (published under the Lewis Carroll pseudonym) since mathematics can be understood as a branch of logic. And it has been claimed, rightly I think, that all of the jokes in his literary works are jokes of pure or applied logic (Braithwaite 1932, 176). Of course, even among academicians Carroll is best known because of *Alice Adventures in Wonderland* and *Through the Looking-Glass*. Thanks to Martin Gardner's *Annotated Alice* (1960), *More Annotated Alice* (1990), and *The Annotated Alice: The Definitive Edition* (2000) and Peter Heath's *The Philosopher's Alice* (1974), though, the philosophical and logical interests of Carroll that permeate his works are fairly widely known.

I hope I didn't mislead by the title of this paper. It is not a paper about squaring the circle, although at one time Carroll started a "little book" that he hoped would refute all those who thought the circle could be squared.[1] In this paper, however, I will compare Carroll's square and rectangular argument-diagrams for categorical arguments to Venn's circular diagrams.[2] But first I want to say a brief word about existential import.

A WORD ABOUT IMPORT

No consideration of categorical arguments would be complete unless it took note of the issue of existential import. A proposition is said to have existential import if that proposition is considered to assert, whether or not it might also make an additional assertion, that the classes referred to have members. The question is whether we should consider our categorical claims to always be about classes with members. Traditionally it is believed that there are only four types of propositions that can be constructed. They are as follows:

- All *x* are *y*, called a universal affirmative or *A* proposition.
- No *x* are *y*, called a universal negative or *E* proposition.
- Some *x* are *y*, called a particular affirmative or *I* proposition.
- Some *x* are not *y*, called a particular negative or *O* proposition.

Carroll's discussion of existential import (as well as diagraming) deals only with the three propositions, "All *x* are *y*," "No *x* are *y*," and "Some *x* are *y*." Carroll immediately and without discussion considers the *O* proposition "Some *x* are not *y*" to be an *I* proposition, "Some *x* are non-*y*." (Carroll writes this as "Some *x* are not-*y*" or "Some *x* are *y′*.")[3] I see no reason to object to this since they are equivalent by obversion and because people often make that obversion easily and naturally without training. The proposition that I want to consider here is the *A* proposition. The obvious, and historical, question is whether it has existential import. The classical interpretation had been that it did assert the existence of class members. And Carroll agrees with this. Essentially the heart of his argument for this is that since *I* propositions assert, and since an *A* proposition "necessarily *contains* a Proposition in *I*" it must also assert the existence of members (Carroll, *Symbolic Logic* 1958, 166).

In describing Venn's system Carroll shows the following diagram (fig. 1) as the way Venn diagrams "All *x* are *y*." (Entities with a certain property constitute a set, described by a "term," represented by a circle, with those particular entities visualized as residing inside that circle. Two circles are necessary to diagram a categorical statement since both the *x* and *y*, above, represent a term. Shading indicates non-membership or an empty cell, while an asterisk, *x*, or cross indicates membership.) This would indicate the existence of members of both *x* and *y*, and would be in agreement with Carroll's own view.

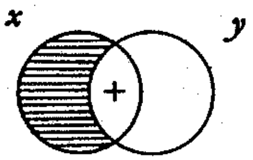

Figure 1

However, the difficulties that Carroll's system of interpretation introduces should not be ignored. Every proponent of the common contemporary interpretation, giving *I* and *O* propositions existential import but withholding it from *A* and *E* propositions, points out that there are many statements of an *A* form for which the speaker does not intend an existential claim. Consider the scientist's "All freely-falling bodies have universal acceleration" or the property-owner's "All trespassers will be prosecuted," or the professor's "Every student averaging at least 90 will receive an A." So it is pretty obvious that Carroll's argument (that an *A* asserts since it necessarily contains an *I*) begs the question since the issue is precisely whether it does contain an *I* proposition.

Furthermore, John Venn (1971) does *not* agree with Carroll's interpretation. He diagrams an *A* proposition (fig. 2) without the existence of members asserted (122):

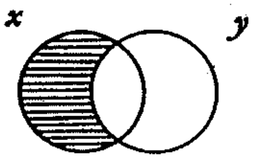

Figure 2

And he says "it is not the existence of the subject or the predicate (in affirmation) which is implied, but the non-existence of any subject which does *not* possess the predicate . . ." (157-58).

So the situation appears to be that Carroll gives existential import to *A* statements and he erroneously says that Venn does the same. Interestingly, in his "Notes to Appendix" a few pages after he has argued for his interpretation, claiming that his is the only system that is not logically inconsistent or involve "great practical inconvenience," he does concede that

> [a]nother view is, that the Proposition "All *x* are *y*" *sometimes* implies the actual *existence* of *x*, and *sometimes* does *not* imply it; and that we cannot tell, without having it in *concrete* form, *which* interpretation we are to give to it. This view is, I think, strongly supported by common usage . . . . (Carroll, *Symbolic Logic* 1958, 171.)[4]

John Venn (1971) had said something similar in a footnote.

> It may be pointed out here, once for all, that I am not proposing to lay it down as a law that no [universal] proposition can have these positive implications [asserting members], but merely to maintain that in the Symbolic Logic, as a means to generalization, they should not be regarded as any part of the proposition. Admit them by all means wherever desired; but as explicit implication, to be distinctly indicated. (158.)

Both Carroll and Venn are aware that speakers sometimes intend for propositions in *A* to assert membership and sometimes they do not have that intention. But Carroll's choice in his system of diagraming was to interpret every *A* as having existential import. Otherwise, he says, "the difficulties, which it introduces, seem to me too formidable to be . . . easily intelligible to mere *beginners*" (Carroll, *Symbolic Logic* 1958, 196). Venn's approach is to adopt "as a means to generalization" only what can be held of all propositions of a type, so he takes the universal content of an A, which is the "negative" claim that *xy*' is empty.

In the Notes to Appendix that I quoted from earlier, Carroll says that the common sense view "will be fully discussed in Part II" (ibid.). In his diary entry for August 8, 1896, written while he was writing Part II, he says, "I find I must re-write, in *Symbolic Logic*, the Section on Propositions in *A*" (Green 1954, 2: 528). He never finished the projected parts II and III of *Symbolic Logic* but it is clear he was not satisfied with his stand in Part I. I also think this coincides with his continuing interest in the logic of hypotheticals that he exhibits, for example, in his famous Barber-Shop Paradox.[5]

I believe that Carroll was getting on the right track about this issue and I believe that Venn had it essentially correct. It may present some difficulties to deal with the ambiguity of *A* statements, but those difficulties are not "formidable." It only requires that sometimes we do an additional bit of work. Diagrams can be drawn according to the non-existential interpretation of the universals (*A* and *E*) as Venn does. Most valid syllogisms will be detected this way. But for those syllogisms that do not test valid this way one must go on and ask if the existential interpretation would make a difference. The simplest way to do this to my knowledge is to use Patrick Hurley's (2000) method (268-70). His system does not require re-diagraming the premises with the existential interpretation. He says that we can simply look at the diagram of an argument that tests invalid, then look to see if there is a circle that has three of its four subsections shaded. If so, and if the argument would test valid if there were an asterisk in that subsection, simply ask the empirical question, "Do members of the class represented by that circle exist." If so, then the argument is valid from the existential standpoint.

ARGUMENT DIAGRAMS

Diagrammatic representation of arguments became common thanks to its employment by Euler and Venn, and almost every student of elementary logic has been exposed to using Venn diagrams to evaluate the validity of arguments. A syllogism with three terms can be represented by three overlapping circles. Witness Venn's diagram and analysis of the following argument:

> No philosophers are conceited.
> Some conceited persons are not gamblers.
> Therefore, some persons who are not gamblers are not philosophers.

If this is rewritten as,

> No *x* are *m*.
> Some *m* are *y*'.
> Therefore, some *y*' are *x*',

then it is easily (Venn) diagramed as follows (fig. 3, using shading to indicate an empty cell) (Carroll, *Symbolic Logic* 1958, 182).

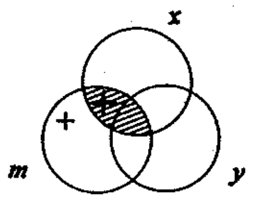

Figure 3

Carroll quotes the following analysis of this very argument, saying "[t]he following Solution has been kindly supplied to me by Mr. Venn himself" (182-83).

- The Minor Premiss declares that some of the constituents in *my*' must be saved: mark these with a cross.
- The Major declares that all *xm* must be destroyed; erase it.
- Then, as some *my*' is to be saved, it must clearly be *my*'*x*'. That is, there must exist *my*'*x*'; or, eliminating *m*, *y*'*x*'. In common phraseology, "Some *y*' are *x*'," or "Some not-gamblers are not-philosophers."

Venn, like many logicians even today, does not represent the universe of discourse, but, as Carroll (*Symbolic Logic* 1958) says, has it "ranging at will through Infinite Space" (176). Although it is easy to enclose Venn diagrams inside a box that represents the universe of discourse, it is interesting that, for example, Copi, Hurley, and Kahane don't. Carroll, on the other hand, prefers using a box to represent every class as well as the universe of discourse. To take an example from Carroll's *The Game of Logic* (1958), "Some new Cakes are unwholesome" first the universe "cakes" is enclosed in a box (21).

Then we may let the top half of the box represent "new cakes" and the bottom half would represent cakes that are not new ones. Then the left side of the original box can then represent wholesome cakes and the right half, cakes that are not wholesome (fig. 4).

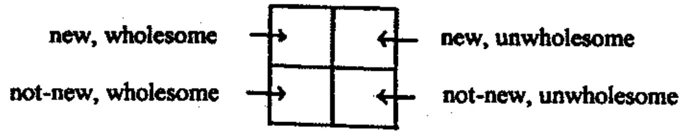

Figure 4

He uses the symbol "1" to represent membership (one or more entities) in the class, and a "0" to show that a region is empty. So, "Some new cakes are unwholesome" becomes (I'll put the equivalent diagram [fig. 5] using shading, for empty cells, and an asterisk, for membership, on the right):

Figure 5

Adding a small square to the center of this diagram allows the addition of a third term. Let's go back to the syllogism above. Diagraming "No *x* are *m*" requires a "0" in the cells that represent *m*s that are within *x* showing *m*s empty (fig. 6). (Let the top of the large square be *x*/philosophers, the left side be *y*/gamblers, and the small internal square be *m*/conceited persons.)

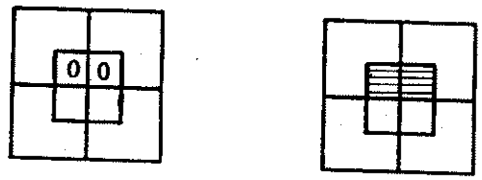

Figure 6

Then we put a "1" in the only remaining possible cell representing *m*s that are *non-y*s (fig. 7). Examination then shows that some non-gamblers are non-philosophers (Some *y*' are *x*'). Thus, the argument is valid.

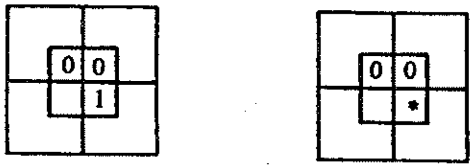

Figure 7

Since the goal of diagraming is developing a system that allows an easy way to assess validity, the question naturally arises at this point whether Carroll's diagrams are easier to use than Venn's. The answer must certainly depend upon who is asked, but I think Venn diagrams are the easier; I am also well aware that may be because I was familiar with them for many, many years before I encountered Carroll's diagrams. I have been able to teach Carroll's diagrams to students with success. On the other hand, it may mean something that the only textbook that I am aware of that uses Carroll's diagrams is Peter Geach's *Reason and Argument* (1976).

But even *if* Venn diagrams are easier for three-term categorical syllogisms, the answer might be different for arguments with more terms. Geach says, "It is hard to draw four-term Venn diagrams (impossible with overlapping *circles*), but quite easy to draw four-term Lewis Carroll diagrams . . . " (59). To draw such Venn diagrams requires ellipses and, quite similarly, drawing such Carroll diagrams requires using some rectangles. Here would be the diagrams (fig. 8) for a simple sorites with four terms (using the convention of shading for an empty cell):

All *w* are *y*.
All *y* are z.
All z are *x*.
Therefore, all *w* are *x*.

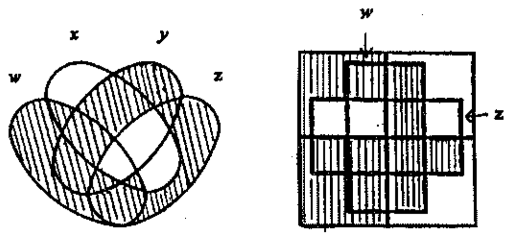

Figure 8

Again, if anything, it seems like the Venn diagram is as simple; but you can judge for yourself.

But the advantage, I think, to Carroll's system is that for any argument with five, six, or seven terms (and even eight, nine, or ten) seems clearly superior. For an

argument with five terms, Venn supplies the following (fig. 9):

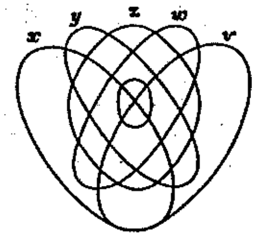

Figure 9

And it needs to be noted that "the small ellipse in the centre is to be regarded as a portion of the *outside* of *z*; i.e. its four component portions are inside *y* and *w* but are no part of *z*" (Venn 1971, 117). Even Venn is forced to say "[i]t must be admitted that such a diagram is not quite so simple to draw as one might wish it to be." This does not bother him since he believes that the only alternative is "nothing short of the disagreeable task of writing out, or in some way putting before us, all the 32 combinations involved." I will show you Carroll's diagram for five-termed arguments just a little later.

The way Venn suggests to handle six terms is to use two copies of his five-term diagram, letting one represent being in *f* and one being outside the *f* class. The problem is that the area outside the two diagrams "ranging at will through Infinite Space" will, even if confined inside a box, have to be shared both by *a'b'c'd'e'f* and *a'b'c'd'e'f'*. This means that the diagram is inadequate since only 63 of the required 64 distinct areas that should be shown on a diagram are represented.

The situation continues to deteriorate for Venn since he does not attempt to go beyond six terms with his diagrams. Carroll, however, presents diagrams for as many as ten terms.

Carroll first (in 1888) devised a five-term diagram with a crooked line running through all the previous cells for the fifth term. I will show this diagram along with the one he developed later for publication in *Symbolic Logic* (fig. 10). The diagram without the crooked boundary line is more in keeping with his diagrams for syllogisms with even more terms. I will work out a few simple arguments with the diagrams. For the sake of simplicity they are done without asserting existential import and with "shading" to indicate empty cells. I will use a different type of shading for each proposition and will not re-shade any cell that has already been shaded. The oblique partitions are used to assign members of *E* to the upper portions and the lower portions to *E'*.

—five terms—
All A are B.
All B are C.
All C are D.
All D are E.
All A are E

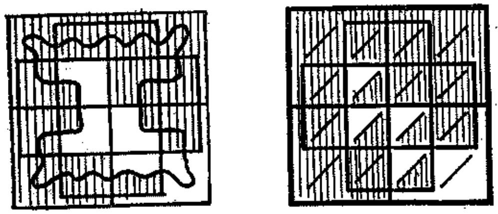

Figure 10

I will add to the premises above "All *E* are *F*," using a sixth term. The conclusion would be "All *A* are *F*." Here Carroll will "substitute upright crosses for the oblique partitions" (*Symbolic Logic* 1958, 177) in each of the previous sixteen cells, and he will assign *E* to the upper half of each and assign *F* to the left half. So the four classes thus indicated are *EF*, *EF'*, *E'F*, and E'F'. Here is a diagram of the argument above but with the additional premise "All *E* are *F*." The conclusion "All *A* are *F*" will be seen to follow (fig. 11).

All E are F.

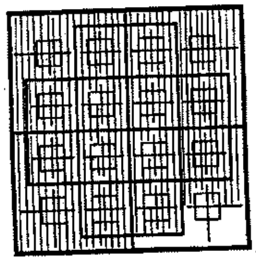

Figure 11

The diagram for seven terms adds "to each upright cross, a little inner square" (ibid., 178). The area inside that square represents the seventh term, *G*. I will diagram an argument like the one above but add the additional premise "All *F* are *G*." The conclusion can be seen to be "All *A* are *G*."

All F are G.

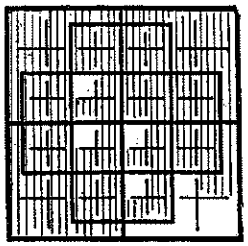

Figure 12

I will not work any more arguments out, but I will explain how Carroll diagrams even larger arguments. For eight letters Carroll places "in each of the 16 Cells, a *lattice*,

which is a reduced copy of the whole Diagram (fig. 13), and just as the 16 large Cells of the whole Diagram are assigned to the 16 Classes [*ABCD*, *ABCD'*, and so on], so the 16 little Cells of each lattice are assigned to the 16 little Classes [EFGH, *EFGH'*, and so on]" (ibid.).

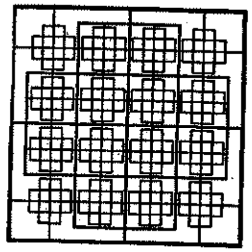

Figure 13

For nine letters Carroll uses two of the eight-letter diagrams side by side and lets one be a member of the added class and the other not a member. For ten terms he uses four diagrams that represent the four combinations of the last two terms.

What, then, does this head-to-head comparison of the two types of diagrams show? It shows that there are no disadvantages to Carroll's system and some clear advantages. These advantages are chiefly for arguments with more than the three terms of traditional syllogisms. If one does not ever want to go beyond the ordinary syllogism and is content to break sorites into a series of syllogisms, the traditional three-circle diagrams are perfectly fine. However, if one desires to work with greater ease with arguments containing more than three terms, one should think about what Carroll has to offer.

Notes

1. The work was to be called *Limits of Circle-Squaring*, and later *Simple Facts about Circle-Squaring*. The few pages that survive can be found in the Lewis Carroll Centenary 1932.

2. John Venn was an English mathematician and logician who lived from 1834 to 1923.

3. For example, when working with the proposition "Some apples are not ripe" his only concern is with "completing" the proposition "by supplying the Substantive 'fruit' in the Predicate, so that it would be "Some apples are not-ripe fruit" (Carroll 1958, 31).

4. He says of the interpretation that "E and A 'assert', but I does not" although it "can logically be held," it involves "great practical inconvenience."

5. This is not the paradox about the barber that shaves all those who don't shave themselves, but is a puzzle about hypotheticals set in a barbershop.

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