Socratic Dialectic And The Analytic Method Of The Ancient Geometers

Michael D. Capistran

In this paper I intend to show that Socratic dialectic in Plato uses the same method of analysis and discovery as that used in ancient Greek geometry.

Part I: The Analysis of the Ancient Geometers

In Euclid, propositions are presented and then given demonstration. This method was recognized as pedagogically deficient at about the time of the late Renaissance.[1] At that time it was recognized that for the student to be simply handed the fully completed proofs of a geometer such as Euclid was, as a method of instruction, inferior; and that for educational purposes it was more important to challenge the student by posing a problem and by then having the student try to solve it. The student learned better by being challenged with a problem to be solved rather than by memorizing the completed solution by rote. This intuition has been carried over into our own modern curriculum in mathematics. Sometimes the problem posed to the student is sufficiently simple to where the student can follow out a line of reasoning similar to the Euclidean exposition and set forth the demonstration immediately. Such demonstrations in Euclidean form are today commonly considered "synthetic" in nature. Often, however, the problem is too difficult to be solved in this fashion, and the student is then required to utilize a different method; a method, as opposed to the standard Euclidean exposition, commonly referred to now as "analytic." There is a sense in which this "analytic" treatment might be characterized as "reasoning backward."[2] It is a method familiar to all of us; it is the method which we all utilized to learn plane geometry in high school. Let me give the following illustration:

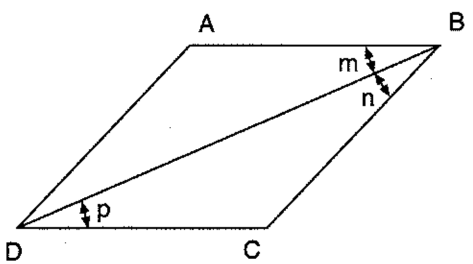

figure 1.

Given that ABCD is a rhombus[3] and BD is a diagonal. Prove that $\angle m = \angle n$.

Solution. First we begin by assuming what we are trying to show: that $\angle m = \angle n$. Now AB $\parallel$ DC, since the figure is a rhombus, and consequently, $\angle m = \angle p$. Therefore, $\angle p = \angle n$ since we assumed that $\angle m = \angle n$; but if $\angle p = \angle n$, then DC must equal BC. We have now arrived at a known or a given, for we know that DC = BC since the figure is a rhombus. The backward movement is now ended and we are ready to give the final proof by reversing our steps as follows:

DC = BC since the figure is a rhombus: therefore triangle DCB is isosceles and $\angle p = \angle n$; but $\angle p = \angle m$ since AB $\parallel$ DC; therefore $\angle m = \angle n$.[4]

This example is taken from a student introductory text to problems in plane geometry, and I have amended it slightly to help elucidate the method involved. I also think it worthwhile to quote verbatim the explanation of the example offered there to the student:

> [We] assume the conclusion and then draw deductions from the conclusion until we arrive at something known or something which can be easily proved. In other words this method is the opposite of the forward movement in which we begin with the given and proceed forward to the conclusion; in the backward movement we begin with the conclusion and move backward till we arrive at the given or the known. After we arrive at the given or the known, we then reverse the movement and proceed forward to the conclusion.
>
> This method is sometimes called the *analytic method.* It is very important and deserves the student's closest study.[5]

That is the situation from a modern perspective. Ancient geometers also adhered to a distinction they took to exist between the two methods of analysis and synthesis for the solution to a problem. Euclid was the post-Aristotelian geometer who, in his work *The Elements*, compiled and collated the different fundamental aspects of geometry into the form with which we are familiar today. In reference to analysis and synthesis, there is an interpolation in the early part of the last book of this work:

> Analysis is an assumption of that which is sought as if it were admitted and the passage through its consequences to something admitted to be true.

Synthesis is an assumption of that which is admitted and the passage through its consequences to the finishing or attainment of what is sought.[6]

Although these statements sit well with our example of the rhombus earlier, they are not in themselves very revealing. From this passage we may say that for the ancient geometers analysis is reasoning backward from that which is sought to that which is admitted, and synthesis is reasoning forward from that which is admitted to that which is sought. Pappas, one of the last great Greek geometers, has something of a fuller account, however. Concerning analysis specifically he has some additional comments, some of which are germane to our discussion:

> In *theoretical* analysis we assume what is sought as if it were existent and true, after which we pass through its successive consequences, as if they too were true and established by virtue of our hypothesis, to something admitted: then (a), if that something admitted is true, that which is sought will also be true and the proof will correspond in the reverse order to the analysis, but *(b)*, if we come upon something admittedly false, that which is sought will also be false.[7]

In the words of Heath:

> The method is as follows. It is required, let us say, to prove that a certain proposition A is true. We assume as a hypothesis that A is true and, starting from this we find that, if A is true, a certain other proposition B is true; if B is true, then C; and so on until we arrive at a proposition K which is *admittedly* true. The object of the method is to enable us to infer, in the reverse order, that, since K is true, the proposition A originally assumed is true. Now Aristotle had already made it clear that false hypotheses might lead to a conclusion which is true. There is therefore a possibility of error unless a certain precaution is taken. While, for example, B may be a necessary consequence of A, it may happen that A is not a necessary consequence of B. Thus, in order that the reverse inference from the truth of K that A is true may be logically justified, it is necessary that each step in the chain of inferences should be unconditionally convertible. As a matter of fact, a very large number of theorems in elementary geometry are unconditionally convertible, so that in practice the difficulty in securing that the successive steps shall be convertible is not so great as might be supposed. But care is always necessary.[8]

Simply put, with analysis we reason backward from our hypothesis until

either (a) we reach a theorem or postulate admitted to be true, in which case the proof or synthesis will correspond in reverse order to the analysis, or (b) we find a contradiction between the hypothesis and the admitted, in which case the hypothesis, the theorem or construction in question, has been shown to be incorrect. This second type of theoretical analysis is called today *reductio ad absurdum* in which a correct inference from a hypothetical proposition leads to a conclusion either (1) admittedly false or (2) contradictory to the hypothesis itself or to some one of its consequences. In this case we can conclude without further inquiry that the presumed hypothesis is false. That is, we postulate a proposition and the theorems of the system. Because we feel we have reason to pronounce the theorems correct we are therefore constrained to pronounce the postulated proposition as incorrect, and have therefore disproven it. Thus, an indirect proof or *reductio*, though not the only form of reasoning backward from the ancient Greek point of view, would nevertheless still be an example of analysis. Again in the words of Heath:

> In the process of analysis starting from the hypothesis that a proposition A is true and passing through B, C ... as successive consequences we may arrive at a proposition K which, instead of being admittedly true, is either admittedly false or the contradictory of the original hypothesis A or of some one or more of the propositions B, C ... intermediate between A and K. Now correct inference from a true proposition cannot lead to a false proposition; and in this case therefore we may at once conclude, without any inquiry whether the various steps in the argument are convertible or not, that the hypothesis A is false, for, if it were true, all the consequences correctly inferred from it should be true and no incompatibility could arise. This method of proving that a given hypothesis is *false* furnishes an indirect method of proving that a given hypothesis A is *true*, since we have only to take the *contradictory* of A and to prove that it is false. This is the method of *reductio ad absurdum*, which is therefore a variety of analysis.[9]

In summation, the ancients saw analysis as reasoning backward and synthesis as reasoning forward. When we start with what has either been originally assumed or already proven and prove a new theorem directly we are using synthesis. When we start with the theorem to be proven and reason backward to what has already been assumed or proven we are using analysis.[10]

Part II: Geometric Analysis in the *Meno*

Consider Socrates' discussion with the servant boy in the *Meno*.[11] There, the problem set by Socrates is that of doubling the area of a given square. That is, a 2' by 2' square is drawn and the servant boy is asked what the length of the side of a square would be which would have double the area of the given square. The boy responds that the *side* of such a square would be double the *side* of the original – since the *area* is to be double. Socrates, by means of performing a *reductio ad absurdum* shows the boy that this answer can't be correct. This is, following out the consequences of the suggestion we find that the area of such a square would be 16', which is not equal to 8'. Socrates also points out that whereas the original line of 2' would obviously be too short, the line suggested by the boy of double that or 4' would be too long. On the basis of this information, the boy then suggests the length of 3' which lies between the minimum length of 2' and the maximum of 4'. Socrates again performs a *reductio* showing the boy that he is wrong – because 9' is not equal to the sought for 8'. At this point the boy is just stumped, and it is pointed out that the boy is in the position of having experienced Socratic *elenchus*. Socrates then calls the boy's attention to the diagonal of the original square. Again the consequences of this suggestion are followed out, but this time the suggested length is seen to be correct for the required doubling of the area.

And here we recognize the method of Platonic dialectic. In the words of Cornford:

> 'Dialectical' has some implications which may escape the modern reader. He will readily understand that dialectic means a co-operative inquiry carried on in conversation between two or more minds that are equally bent, not on getting the better of the argument, but on arriving at the truth. A tentative suggestion (*'hypothesis'*) put forward by one speaker is corrected and improved until the full meaning is clearly stated. The criticism that follows may end in complete rejection or lead on to another suggestion which (if the examination has been skillfully conducted) ought to approach nearer to the truth.[12]

We recognize in this characterization Socrates' quiet, chiding counter-examples against Charmides' conjecture that temperance is quietness;[13] we see in it the thrashing of Thrasymachus' view that justice is the advantage of the stronger,[14] and of Euthyphro's claim that the holy is that which is loved by the gods;[15] also we perceive it in Socrates' defense against Meletus' contradiction that Socrates is guilty of not believing in the gods, but believing in the gods.[16]

The method in question is made reasonably explicit in the *Phaedo* where Socrates, addressing Cebes, states:

> ... I started off in this way, and in every case I first lay down the theory which I judge to be soundest, and then whatever seems to agree with it – with regard either to causes or to anything else – I assume to be true, and whatever does not I assume not to be true ....[17]

That is, Socrates first suggests a hypothesis and then follows out the logical consequences of the hypothesis.[18] Socrates continues:

> So you, too, [Cebes] like myself [would do the same],
> ... while you [like myself], being nervous of your own shadow, as the saying is, and of your inexperience, would hold fast to the security of your hypothesis and make your answers accordingly. If anyone should fasten upon the hypothesis itself, you would disregard him and refuse to answer until you could consider whether its consequences were mutually consistent or not. And when you had to substantiate the hypothesis itself, you would proceed in the same way, assuming whatever more ultimate hypothesis commended itself most to you, until you reached one which was satisfactory. You would not mix the two things together by discussing both the principle and its consequences, like one of these destructive critics – that is, if you wanted to discover any part of the truth ... you, I imagine, if you are a philosopher, will follow the course which I describe.[19]

There thus seem to be two distinct steps to the process put forward by Plato. First the conjecture of an opinion to be tested; and next the testing of the opinion.

When asking the question, "What is knowledge?" in the *Theaetetus* Socrates speaks of the same method in terms of getting a ball through a hoop:

> I cannot make out to my own satisfaction what knowledge is. Can we answer that question? What do you all say? Which of us will speak first? Everyone who misses shall 'sit down and be donkey,' as children say when they are playing at ball; anyone who gets through without missing shall be king and have the right to make us answer any question he likes.[20]

Part III: The Dimension of Discovery in Analysis

Proclus has associated analysis with discovery, claiming that Leodamas used this method as taught to him by Plato to discover many things in geometry.[21] Let us investigate this association of geometrical analysis or dialectic with discovery in Plato.

Consider again Socrates' discussion in the *Meno* with the servant boy. Recall that Plato has included this example taken from geometry because he feels it is somehow elucidatory. Let us recapitulate again what actually happens. The servant boy suggests an answer; this answer is shown to be wrong. The servant boy then suggests a second answer, closer than the first to being correct, but one still shown by Socrates to be wrong. At this point the *elenchus* occurs.

And it is also at this point in the example we may consider the manner in which a discovery is actually made – not just in geometry, but of any kind. Consider the following statement on the part of the mathematician Polya:

> The coming of a bright idea is an experience familiar to everybody but difficult to describe and so it may be interesting to notice that a very suggestive description of it has been incidentally given by an authority as old as Aristotle.
>
> Most people will agree that conceiving a bright idea is an "act of sagacity." Aristotle defines "sagacity" as follows: "Sagacity is a hitting by guess upon the essential connection in an inappreciable time. As for example, if ... observing that the bright side of the moon is always toward the sun, you may suddenly perceive why this is: namely, because the moon shines by the light of the sun."[22]
>
> We should realize that a contemporary of Aristotle ... saw the full moon as a flat disc, similar to the disc of the sun but much less bright. He must have wondered at the incessant changes in the shape and position of the moon. He observed the moon occasionally also at daytime, about sunrise or sunset, and found out "that the bright side of the moon is always toward the sun" which was in itself a respectable achievement. And now he perceives that the varying aspects of the moon are like the various aspects of a ball which is illuminated from one side so that one half of it is shiny and the other half dark. He conceives the sun and the moon not as flat discs but as round bodies, one giving and the other receiving the light. He understands the essential connection, he rearranges his former conceptions instantly, "in an inappreciable time": there is a sudden leap of the imagination, a bright idea, a flash of genius.[23]

This is Socrates' next move when he indicates to the servant boy the diagonal of the square for his consideration. We observe that the diagonal cuts the square in half, and now we may perceive, "in an inappreciable time," with a sudden leap of the imagination, that the side of the square which quadruples this area is the side sought after.

But Socrates does more than just challenge the boy. Recall that this example in the *Meno* is offered by Plato as an example of *anamnesis*. Again, let us consider

the manner in which a discovery, any discovery, is actually made. First, we familiarize ourselves with the problem and especially with the unknown in the problem. Next we propose a solution. Then we test the proposed solution. If we find any contradiction, we throw the proposed solution out. This is the *reductio*. We are better off for having discontinued the proposition, however, because this helps us focus on a solution. When the solution occurs, it not uncharacteristically happens as a flash of insight or enlightenment. In our case of the servant boy, however, the boy has no more than true opinion because he has followed the logical sequence of the proof without having understood the reason why the diagonal is the correct length. Someone with knowledge would understand the reason why, or the *logos*. That is, the person with knowledge would perceive, as a sort of gestalt switch, that the figure enclosed by the four diagonals is an area precisely twice the original square.[24]

Or, to use a different example, suppose we were to discover a simple proof for, say Fermat's last conjecture. Whence does the discovery derive? Such enlightenment was traditionally considered a gift from the gods, a function performed here for the servant boy by Socrates acting as midwife. Plato seems to have believed that knowledge is already inherent, sleeping as it were, within us and the recognition of this knowledge is the origin of the discovery. Initially speaking, for people who actually make discoveries – and are amazed at themselves – this is not perhaps an unreasonable first conjecture. When we are searching for the answer, we are searching for something we believe surely and actually exists. Why can one person so readily solve a problem when others cannot? What is meant by the term "insight"? Since the information does not derive from the conscious mind, it is entirely believable that it derives from somewhere . . . else. Hence the thesis of *anamnesis*. The situation is somewhat similar to that of dream. I am the one in my dream; but who is the one setting up and manipulating the dream? It is not me in the same sense. All this seems to come from somewhere . . . else.

In summation, then, analysis or reasoning backward was the method used by the ancient Greek geometers to discover the proof of their theorems and was also the method emulated by Plato. It is not only a way of showing some conjecture is wrong; it is also considered by Plato a method of discovery, and it is this method Socrates characteristically uses in his attempts to discover the nature of the various virtues. Such is the method of Socratic Dialectic.

Notes

1. Vide: Howell, W.S., *Logic and Rhetoric in England, 1500-1700*, Princeton, 1956; Gilbert, N.W., *Renaissance Concepts of Method*, New York, 1960; and especially Ong, W.J., *Ramus, Method, and the Decay of Dialectic*, Cambridge, Mass., 1958.

2. An example of reasoning backward for the ancient Greeks was the reduction by Hippocrates of the problem of doubling the cube to that of finding two mean proportionals. Vide: Scott, J.F., *The Scientific Work of René Descartes*, Taylor & Francis, Ltd., London, 1952, p. 94; Scott in turn refers to Plato, *Meno: 86e-87a*.

3. Rhombus: A quadrilateral with four equal sides.

The additional property we need to know to perform this proof is that the opposite sides of a rhombus are parallel.

4. Horblit, Marcus & Kai L. Nielsen, *Plane Geometry Problems with Solutions*, Barnes & Noble, Inc. New York. 1947, pp. 6-7.

5. Ibid, p. 6 & p. 8.

6. Heath, Thomas, *Euclid: The Thirteen Books of The Elements*, 3 vols., 2nd ed., Dover, New York, 1956 (reprinted from Cambridge University Press, 1926), v. 1 (referred to herein as: Heath, *Euclid I), p. 138.*

This interpolation appears in every manuscript we have of the *Elements*, and in some has even entered from the margin into the body of the text. Heath has the following to say concerning the passage: "The interpolation took place before Theon's time, and the probability is that it was originally in the margin, whence it crept into the text of P after XIII.5." Heath, *Euclid III*, p. 442.

7. Heath, *Euclid I*, p. 138. This celebrated passage is worth looking at in its entirety, though it seems inappropriate to include it all here.

8. Heath, *Euclid I*, p. 139.

9. Heath, *Euclid I*, p. 140. Heath has not here appreciated the difference between a *reductio* and an indirect proof.

10. For performing proofs in modern formal logic, we often instruct our students to try reasoning backward in the sense of asking which step will yield the end result sought, and which step will yield the second to last step, and so forth. Thus, we are teaching our students to reason backward, but in terms of *abduction:* Which step is *sufficient* to produce the result sought? From the passages from Euclid and Pappus given above, it is clear the Greeks thought of reasoning backward in terms of *deduction*: What are the *necessary consequences* of the result sought? As both are cases of reasoning backward, and as both require reversal to produce the required synthesis, both should therefore be considered analysis.

I am indebted to Dr. Olin Joynton for helping clarify this issue.

11. Plato, *The Collected Dialogues of Plato*, Edith Hamilton and Huntington Cairns, eds., Bollingen Series LXXI, Princeton University Press, Princeton, New Jersey, 1961 (referred to herein as: Plato), 82b-85b, pp. 365-370.

12. Cornford, Francis, M., *Plato's Theory of Knowledge (the* Theatetus *and the* Sophist *of Plato)*, the Bobbs-Merrill Company, Inc., N.Y., 1957, p. 30. This passage is the closest Cornford comes to associating Platonic dialectic with geometrical analysis.

Taylor gives a similar account when he states: "... the definition ... is, in fact, put forward tentatively as a suggestion for examination. The examination is conducted in strict accord with the requirements of the dialectical method as described in the *Phaedo*. The first step is to see what consequences follow from the suggested "postulate." If the consequences are found to be in accord with known facts, and thus so far "verified," the postulate will be regarded as *so far* justified; if some of them prove to be at variance with fact, it must be modified or dismissed, it cannot hold the field as it stands." Taylor, A.E., *Plato: The Man and his Work*, Meridian Books, Inc., N.Y., 1960. p. 32. This is the closest Taylor comes to the association of dialectic with geometrical analysis, though closeby he says: "The generalization ... has yet to be tested and may have to be rejected. The testing is the work of intellectual analysis, or, as Socrates and Plato call it 'dialectic'." The term 'analysis' is given no specific meaning here.

13. Plato, *Charmides*. 159c-160d, pp. 105-6.

"Well then, in all concerns either body or soul, swiftness and activity are clearly better than slowness and quietness?"

"Probably."

14. Plato, *Republic*, 339d, p. 589.

"Then on your theory it is just not only to do what is to the advantage of the stronger but also the opposite, what is not to his advantage."

15. Plato, *Euthyphro*, 10d, p. 179.

"So what is pleasing to the gods is not the same as what is holy, Euthyphro, nor, according to your statement, is the holy the same as what is pleasing to the gods. They are two different things."

16. Plato, *Apology*, 26c, p. 12, & 27a, p. 13.

"It certainly seems to me that he is contradicting himself in this indictment, which might just as well run: Socrates is guilty of not believing in the gods, but believing in the gods."

17. Plato, *Phaedo*, 100a & e, 101d, p. 81.

18. Reasoning backward or analysis is not only deductive in character – as I have suggested so far and as indicated in Pappus and in the interpolation in Euclid; reasoning backward may also be inductive. The search may be for sufficient conditions as opposed to the search for necessary and sufficient conditions. To demonstrate the correctness of the proposed hypothesis, however, the steps must then be reversed and the hypothesis demonstrated from the given. For example in the case of the rhombus given above we may ask what the sufficient conditions are for the production of the theorem in question and then reverse the steps to prove the theorem rather than reason backward exclusively in terms of necessary and sufficient conditions. Both cases would be examples of analysis or reasoning backward. I am indebted to Dr. Olin Joynton for helping me clarify certain aspects of the difference.

19. Plato, *Phaedo*, 100a & e, 101d, pp. 82-3.

20. Plato, *Theaetetus*, 146a, p. 850.

21. The method called analysis is associated with Plato by both Proclus and Diogenes Laertius. Vide: Heath, Sir Thomas, *A History of Greek Mathematics. Vol. I*, Dover Publications, Inc., New York (referred to herein as: Heath, *History I*), p. 291.

Heath, however, does not associate Platonic dialectic with geometric analysis as is clear from his statement: "But analysis being according to the ancient view nothing more than a series of successive reductions of a theorem or problem till it is finally reduced to a theorem or problem already known, it is difficult to see in what Plato's supposed discovery could have consisted." Heath, *History I*, p. 291.

22. Polya's note: "The test is slightly rearranged. For more exact translation see William Whewell, *The Philosophy of the Inductive Sciences* (1847), vol. II, p. 131."

23. Polya, G., *How to Solve It: A New Aspect of Mathematical Method. 2nd edition,* Princeton University Press, Princeton, New Jersey, 1945, 1957, 1971 (referred to herein as: Polya) pp. 58-9.

24. Polya points out a differentiation between what he calls, "problems to prove," and "problems to find." Briefly, within the context of the example given here by Plato, the backward movement of the connecting linkage between the hypothesis that the side sought is the diagonal and the accepted truths of geometry and arithmetic is a "problem to prove," whereas the appreciation of the diagonal as a worthwhile candidate for hypothetical test is a "problem to find." Both, however, may be considered reasoning backward in the sense that both start out by asking the question, "What are the sufficient conditions for the possibility of the solution of the problem at hand?" A problem to prove must resolve the tension between the given and the known, whereas the problem to find must resolve the tension between the suggested hypothesis and the known. Both, however, utilize the same analytic or backward approach, and within the context of Greek geometry both may occur, as potentially in this case, simultaneously.

Vide the sections on *Working Backwards* and *Problems to prove/find* in Polya.