

## PEIRCE, TURING AND HILBERT: A SKETCH OF PRAGMATISM VS. FORMALISM

Arthur Franklin Stewart

David Hilbert (1862–1943), the leading German mathematician of his generation, expressed through what he termed "my proof theory" the conviction that all questions in mathematics could be answered by algorithmic means.<sup>1</sup> The Hilbert program of formalism, as it came to be known, counted among the items in its charter the belief that the truth or falsehood of any given mathematical statement could be obtained by some mechanical implementation of fixed rules. The idea of such a definite method employing fixed rules explicitly required each group or listing of such rules or steps to be of *finite length*. Each and every mathematical issue was seen as decidable according to some such definite method. This question of decidability, or the *Entscheidungsproblem*, in its German formulation, was a major component of the Hilbert program, a program which sought to delineate all of mathematics in terms of strictly formal properties. From this point of view, the values of given equational symbols in and of themselves are irrelevant: the patterns that occur are what determine meaning. Addressing this issue of formalism for geometry, Hilbert asserted as early as 1891: "It must be possible to replace in all geometric statements the words *point, line, plane* by *table, chair, beer-mug*" (Reid 1970: 264). This being the case, the powers of empirical observation involved in using points, lines, and planes in visually observable geometric constructions are critically diminished in value, if not rendered entirely superfluous. What matters, instead, is the formal consistency and integrity of the system itself. For Hilbert, then, the statement "table is to chair as chair is to beer-mug" was *formally* equivalent to "point is to line as line is to plane."

This type of anticipated formal equivalence relies explicitly on the principle of *axiomatization*. The primacy of axioms as components of formalized systems of geometry had been in mathematical currency since antiquity:

Ever since Euclid, axiomatizing a theory has meant presenting it by singling out certain propositions and deducing further ones from them; if the presentation is complete, it should be the case that *all* statements which could be asserted in the theory are thus deducible. (Parsons 1967)<sup>2</sup>

Thus, a completely axiomatized system should provide for an investigator to deduce each and every true statement within it. Hilbert entertained no doubts that his own efforts toward an exhaustive formalization of mathematics would not only prove successful in this regard, but that such efforts towards a thoroughgoing axiomatization of the subject could be accomplished with actual *ease*: "Hilbert . . . thought of his programme as one of *tidying up loose ends*" (Hodges 1983: 93, emphasis added). Thus, in 1899 Hilbert went beyond his table, chair, and beer-mug speculations of 1891, constructing an axiomatization of Euclidean geometry that did not rely on references to concrete, visually observable examples in the physical world. With this important development Hilbert had accomplished a major step toward separating abstract, formalized aspects of mathematics from possible empirically derived origins and applications. This success in the construction of his "formula game" committed Hilbert to what was clearly an anti-experimental theory of knowledge:

For this formula game is carried out according to certain definite rules, in which the *technique of our thinking* is expressed. These rules form a closed system that can be discovered and definitively stated. The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds. . . . If any totality of observations and phenomena deserves to be made the object of serious and thorough investigation, it is this one—since, after all, it is a part of the task of science to liberate us from arbitrariness, sentiment and habit and to protect us from the subjectivism that [has] already made itself felt. (Reid, 1970: 185–86).

Such a closed system, one which operates according to a protocol of rules that can be definitively stated is, in its formalized employment of axioms and their treatment by algorithmic means, what I term a "mechanicalistic" or anti-pragmatic epistemology: it abjures considerations of those same *conceivable* effects that are central to Peirce's pragmatic maxim.<sup>3</sup>

Our story resumes with Hilbert's address to the Second International Congress of Mathematicians held in Paris in 1900. His manuscript contained a list of twenty-three questions whose investigation he believed would help define the future course of mathematics.<sup>4</sup> Here Hilbert's intended axiomatization of the whole of mathematics attained a new sophistication by way of specificity. The last of his questions dealt with devising algorithmic means for, as Hopcroft put it, "establishing the truth or falsity of any statement in [the] language of formal logic called the predicate calculus" (1984: 86). Hilbert believed that the answer to this question must necessarily be an affirmative one, for not to be able to employ a strictly algorithmic, deductive apparatus in the establishment of all statements in that calculus would jeopardize the security of his overall system. In 1904, however, his public declarations on foundationalism ceased, not to be resumed until his address to the Swiss Mathematical Society in Zurich during 1917. Here he announced four problems for the foundations issue. The last of these concerned the decidability of a mathematical question by a finite procedure. As Reid suggests, the lecture might as well have been named "In praise of the axiomatic method" (1970: 151). One might say, in view of the demand for a finite, mechanical procedure, it should have been titled "In praise of the *algorithmic* or *mechanicalistic* method." Another eleven years were to elapse before Hilbert stated his aim for the algorithmic solvability of mathematical questions in the form which is of interest to us here.

He chose a most conspicuous venue in which to state this program, namely the 1928 International Congress of Mathematicians held in Bologna. This was the first set of international meetings to which the Germans had been invited since World War I:

At that 1928 congress, Hilbert made his questions

quite precise. First, was mathematics *complete*, in the technical sense that every statement (such as "every integer is the sum of four squares") could either be proved, or disproved. Second, was mathematics *consistent*, in the sense that the statement " $2 + 2 = 5$ " could never be arrived at by a sequence of valid steps of proof. And thirdly, was mathematics *decidable*? By this he meant, did there exist a definite method which could, in principle, be applied to any assertion, and which was guaranteed to produce a correct decision as to whether that assertion was true. (Hodges, 1983: 91)

Hilbert again believed that the answers to these questions *must* be affirmative ones. However, the Czech mathematician Kurt Gödel was to demonstrate, in 1931, that arithmetic, for instance, could not meet the demand for completeness.<sup>5</sup> It is, however, how the question of decidability, or the *Entscheidungsproblem*, was answered by Alan Turing (1912–1954) that is of greater importance for the present analysis.

Turing, after having turned in his dissertation at Cambridge, went on to enroll in an advanced course in the foundations of mathematics under M. H. A. Newman. Newman had heard Hilbert's 1928 address, and in the conduct of his own foundations course brought Turing up-to-date with how the completeness and consistency demands of Hilbert had been handled by Gödel. The question about decidability of mathematical systems in general and arithmetic in particular, however, still remained unanswered. Turing was to demonstrate how this question required a negative answer.

Newman had recast this third question, modifying the thrust of it from decidability as to truth or falsity, to decidability as to provability. We can enunciate this issue as the question, "Is there a purely *mechanical* means for deciding the provability of any proposition X of a mathematical system." For the Hilbert program, as Hodges comments, it was required that the truth of any assertion X "be shown by working *within* the axiomatic system" (1983: 92). Could an axiomatic system show the truth of *every* true mathematical assertion which was contained in it? If so, the system would be complete, and in one sense, then, the *Entscheidungsproblem* would be settled

in the affirmative. If not, then the system would be incomplete and the *Entscheidungsproblem*, in this sense, would be settled in the negative. Of course, Gödel had already disturbed the alleged impeccability of the Hilbert system by showing arithmetic, as a system, to be incomplete:

This was an amazing new turn in the enquiry, for Hilbert had thought of his programme as one of tidying up loose ends. It was upsetting for those who wanted to find in mathematics something that was absolutely perfect and unassailable; and it meant that new questions came into view. . . . [But, the] *third* of Hilbert's questions still remained open. (Hodges, 1983: 93)

This third of Hilbert's questions deals with our second and further sense of the *Entscheidungsproblem*, namely:

Was there a definite method, or as Newman put it, a *mechanical process* which could be applied to a mathematical statement, and which would come up with the answer as to whether it was provable? (Hodges, 1983: 93)

For this further decision problem, it is *not* required that a decision process be effected *within* the system: what is crucial is merely that the decision process be a mechanical or algorithmic one. Thus, whether a proposition is provable or not within the system may be decided by a process that is not itself contained within the system. Moreover, what is being decided upon is the provability of statements and not, at least directly, their truth.

Turing exquisitely demolished the hope that Hilbert's third question could be answered in the affirmative. In doing so, he relied on the notion of Turing machines as an interpretation of mechanical or algorithmic procedures. He employed this notion to prove false the claim that *each and every* statement of a given mathematical system could be *mechanically identified by an algorithmic process to be provable or not*.

What, then, is a Turing machine? According to Turing

himself, each individual Turing machine computes a "computable number" according to its "machine configuration" or algorithm. In his "On Computable Numbers" of 1937 Turing had defined a computable number as a number the expression of which as a decimal is "calculable by finite means." For Newman's version of Hilbert's decidability question to be answered in the affirmative, we should be able to determine, by purely mechanical, finite means, for *each and every* mathematical assertion of *any* given system, whether that assertion is provable or not. This, however, can be shown to be impossible. In order to appreciate why this is impossible, we need to bear in mind an important fact that Gödel had established in his classic work of 1931, namely, that mathematical *propositions* in a mathematical language of the sort Hilbert wished to employ can be correlated one-to-one with *numbers* (see again, please, note 5, above). The exact technique for establishing this correlation between propositions and numbers, known as "Gödel numbering," is not a subject that requires detailed discussion for the present. What is important is that Gödel numbering not only enables one to express propositions *of* a given mathematical system in terms of numbers, it also enables one to express assertions *about* a given mathematical system in terms of numbers. In this way, then, we can correlate the set of provable propositions of a mathematical system with a unique set of numbers, the Gödel numbers of the provable propositions of that system. For Hilbert's decidability question, then, it follows that the set of provable propositions of a mathematical system in its *entirety* is ascertainable by an algorithmic procedure, that is, ascertainable by a machine, if and only if its correlated set of Gödel numbers is decidable in its entirety by an algorithmic or machine procedure.

Turing showed that there are indeed numbers that are not machine computable, even when we understand by "machine" a machine in the broadest possible sense: a Universal Turing Machine. Such uncomputable numbers, moreover, include the number whose *n*th figure is 1 if *n* is the Gödel number of a provable proposition of arithmetic, and 0 if *n* is not the Gödel number of a provable proposition of arithmetic. Thus, Turing showed that there is at least one mathematical system the provable propositions of which are

*not* machine-determinable: formal arithmetic. And the production of but *one* such counter-example is sufficient to disprove Hilbert's original hypothesis about the provability of *all* such systems.

In showing his result, Turing employed a version of a famous argument created about fifty years earlier by the German mathematician Georg Cantor (1845–1918). The Cantor "diagonal argument" shows that the rational numbers (in effect, the simple common fractions) cannot be correlated one-to-one with the real numbers in their entirety.<sup>6</sup> Cantor showed that if we have an alleged list of all possible real numbers, and thereby an alleged correlation of the real numbers with the counting numbers or positive integers, we can use this list to construct a real number which is *not* included in the list. This is a contradiction. Turing, in effect, argued analogously about the set of Gödel numbers corresponding to the provable propositions of arithmetic. If we allege that the set of these numbers is machine decidable, then we can arrange them in a list. This list may then be used to construct a Gödel number of a provable proposition of arithmetic, which number, however, is *not* in the list. This is a contradiction, and shows that the provable propositions of arithmetic are not in their *entirety* machine decidable. It is worth considering Turing's argument in a little more detail, by looking at a version of it that is quite similar to Cantor's actual diagonal argument.

Let us propose an array of Turing machines, each machine "matched" to the production of an individual number whose expression as a decimal is calculable by finite means. This proposal accords with Turing's own definitional scheme. On the definition of the Universal Turing Machine, then, *each and every* individual number so calculable, each and every computable number, must be computable by the Universal Machine. Turing's question was, by analogy, can we find a number that is *not* computable according to this definition by computability? That is, if we can find a number whose expression as a decimal is, on this definition of computable numbers, *uncomputable*, we would then—to continue Turing's analogy—have a negative answer to Hilbert's decidability question: not *every* mathematical assertion in a given system could be determined to be provable or not by a purely

mechanical process. Let us now seek out such an uncomputable number.

Our task, now, shall be to illustrate by analogy Turing's technique of *reductio ad absurdum*. Assume we can mechanically compute every real number between 0 and 1 into a decimal expression. This assumption then enables us to construct at least one additional such decimal expression not among those computed. We begin a list of such mechanically computed decimal expressions and continue adding to it. We can also assign a numeric counting series to our list of decimal expressions, thereby demonstrating that, although we can continue adding new decimal expressions *ad infinitum*, the individual members of our list are nevertheless *countable*. We also know, based on the definition of a Turing machine, that the computation of each such decimal expression is accomplished by purely algorithmic, mechanical means. Hence, even though the series of digits representing each individual decimal is, like the list of decimals itself, of infinite length, we can, at any specified time, note to which decimal *place* as well as to which decimal *expression* computation has proceeded:

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1 .50000000000000000000 . . .
2 .33333333333333333333 . . .
3 .25000000000000000000 . . .
4 .66666666666666666666 . . .
5 .20000000000000000000 . . .
6 .16666666666666666666 . . .
7 .40000000000000000000 . . .
8 .75000000000000000000 . . .
9 .1428571428571428571 . . .
10 .60000000000000000000 . . .
11 .12500000000000000000 . . .

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(Hodges 1983, 101)

Now consider the underlined diagonal string of digits, which reads 53060600200. . . . Change this diagonal number by adding 1 to each digit, with the exception that 9 converts to 0: 64171711311. . . . Construction of this infinite decimal, used

some fifty years previous to Turing's demonstration by Cantor to display the existence of irrational numbers, is also a method for identifying an uncomputable number. This number *cannot* be a computable number, because it differs from the first computed decimal number in its first decimal place, from the fifth computed decimal in its fifth place, and will, in like fashion, *always* differ from a given computable number. Hence, this number cannot *ever* be included in the list of mechanically computable numbers, no matter to what length the list is extended. On Hilbert's assumption, this diagonal number, as a computable number by Turing's definition, had to be included in the list, and yet was later shown by Turing himself *not* to be in the list: thus, a glaring contradiction. This diagonal number, by virtue of being new, or not in "the list," is accounted for by a creative, pragmatic act of observation and learning rather than by a non-creative, mechanical, anti-pragmatic operation. To this number, then, would correspond a provable mathematical assertion which cannot be demonstrated to be such by those same mechanical means that Hilbert's formalism demanded as exhaustive: again, a contradiction. With these "conceivable practical effects," then, the Hilbert decidability question is settled in the negative, and his anti-pragmatic program for epistemology is brought into question.

## NOTES

<sup>1</sup>See Reid 1970 and Hodges 1983. Reid's book, arguably one of the better expositions on Hilbert in English, nevertheless suffers from a pronounced lacuna of references. No bibliography is supplied, a cursory group of notes is given for the entire work, and the index is but a rudimentary compilation of names. This situation thwarts thorough documentation of the many telling remarks by Hilbert which Reid supplies. I therefore enjoin the caveat given by Hodges at his note 2.9, which states, "Here and elsewhere I have drawn upon C. Reid, *Hilbert* . . . for quotations."

<sup>2</sup>Peirce's own definition of "axiom," given in the original, 1889 edition of *The Century Dictionary and Cyclopaedia*, comports well with Parsons' remarks. In the course of his definition, Peirce likewise outlines but further

details the history of this term, referring in due course to Plato, Aristotle, Euclid, Leibnitz, Kant, Bacon, Zeno, and "modern mathematicians" (preserved at P 00373 in Ketner and Stewart 1986).

<sup>3</sup>The version of this maxim Peirce provided during the course of his "How to Make Our Ideas Clear" (*Popular Science Monthly* 12 (1878) reads: "Consider what effects, which might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object."

<sup>4</sup>At the urging of his colleagues Oskar Minkowski and Adolf Hurwitz, Hilbert shortened his remarks by actually stating only ten of the twenty-three questions he had prepared: nos. 1, 2, 6, 7, 8, 13, 16, 19, 21, and 22. The text of Hilbert's address, translated with his permission, was preserved by Dr. Mary Winston Newson in the *Bulletin of the American Mathematical Society* 8 (1902). Apparently Minkowski was of the opinion that the formation of this list of "foundations" questions for mathematics would ensure Hilbert's position in the history of mathematics: he was right.

<sup>5</sup>See Gödel's "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I," *Monatshefte für Mathematik und Physik* 38 (1931): 173–98, preserved as "On Formally Undecidable Propositions of Principia Mathematica and Related Systems I," in Davis 1965.

<sup>6</sup>More complete definitions of the important terms for this example are given under the appropriate entries in Baker 1961. A rational number is a number which can be expressed in the form  $p/q$  where  $p$  and  $q$  are integers and  $q$  is not zero ( $1/2$ ,  $1/3$ ,  $1/4$ , and so forth). An irrational number is a number which cannot be so expressed as an integer or the quotient of integers ( $\pi$ , the square root of 2, and the like). A real number may be rational or irrational.

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