CONSTRUCTIBILITY IN MATHEMATICS: A COUNTERFACTUAL APPROACH TO NON-EUCLIDEAN GEOMETRIES IN KANT'S PHILOSOPHY OF IMAGINATION

Luciana Garbayo
University of Texas at El Paso

Winner of the Houghton Dalrymple Award

I. Introduction

Imagine this: the non-Euclidean geometer lad Bolyai and the old and wise Kant strolling together along the Königsberg Bridges on a very nice day and talking about, of course, the astounding advancements of non-Euclidean geometries. Bolyai is extremely excited, Kant, extremely surprised. In their sincere dialogue lies the desire to build a philosophical bridge between them. As Kant takes in such incredible news, Bolyai candidly asks: "Would there be a way, within Kant's work, to allow for such mathematical novelties to take place, without having to dismiss it all, as mainstream philosophical opinion would take to be unavoidable?"[1] Kant pauses, takes a couple of minutes, and thinks of how he once struggled with the problem of imagination in mathematics, when he decided to restrict it at first to the realm of the rule-following activity of the understanding and gave it no creative ground—simply to restrict it to the pure conditions of sensibility, as necessary conditions of geometry. Yet, the elderly Kant also reminded himself of his last big realization, in the *Critique of Judgment*, where finally, imagination is paired with the activity of reason and with reflexive judgments. Such pairing could perhaps compensate for the strict anchoring of his geometry to sensible intuition, unavoidably pinned to local grounds, as he now realizes (and his follower, polymath Poincaré, would had told him later on). "From the viewpoint of my late work, there might be ways to address different geometrical worlds," Kant considers, "if imagination could also have additional freedom to address universals, beyond the activity of the understanding and sensibility." But there is only so much one also can take up, in being updated with the novelties in mathematics at first. Bolyai will have to give Kant a good summary of what happened in the major areas of mathematics, so that such a project can be successful, and change and continuity in mathematics can be seen and thought through Kantian eyes, too.

This paper is inspired by such apocryphal story. Had Kant had the opportunity to be updated on non-Euclidean geometries, what else would we have inherited from the critical project, regarding the role of imagination in mathematics? As known, the contemporary picture of imagination is quite different today. Luminaries such as Hilbert have said that "mathematics requires more imagination than literary fiction."[2] Here imagination is not restricted, as in the Kantian picture, but is hinted to be a faculty that demands invention, and counter-intuitively so. The distinguished cognitive neuroscientist Dehaene, in his version of the history of number theory (1997: 87-88), focuses on the distinction between the invention of mathematical notions and the importance of finding a sensible intuitive representation to make something new acceptable and communicable. He cites the case of Cardano's discovery of complex numbers in 1545, an extremely important event and yet his results were dismissed by Descartes as being "imaginary numbers"—which was not meant as a compliment, in Cartesian terms, as we all well know—and by De Morgan, who said they were "devoid of meaning." This situation changed, finally, when the British mathematician John Wallis, more than a century later in 1685, found a way to express those numbers through a concrete representation, in diagrammatic form.[3] As for a "moral" behind this little story of struggle "to see" the point of a mathematical discovery, Dehaene suggests that such difficulties arise, among other things, because "to function in intuitive mode, our brains need images—and as far as number theory is concerned, evolution has endowed us with an intuitive picture only of positive integers" (ibid). The same reasoning applies to non-Euclidean geometries. After Bolyai, Gauss and Lobachevski realized the implications of not having a proof to the fifth postulate derived from the other Euclidean axioms, it took years for a representation of hyperbolic space to emerge, as later were produced independently by Beltrami and by Poincaré.[4]

Here I would like to address my Kant-Bolyai story, and take seriously Hilbert's hint regarding the need to consider a larger scope for imagination in mathematics. Hence I suggest a revision of Kant's account of imagination in line with Makkreel's *Imagination and Interpretation in Kant* (1990) towards a theory of interpretation, which I consider might make possible to deal with non-Euclidean geometries, within his approach. That would be the case, I suggest, by taking up the following problems devised by Parsons (1984): (1) the absence in Kant of a "stable" theory of number, in relation to the absence of a theory of mathematical objects in general, and (2) the limitations sprung from the strict association to each other of symbolic and ostensive construction, for effectively understanding concepts in algebra and arithmetic.

My main claim is that Non-Euclidean geometries could indeed have been thinkable for Kant if he had explored these problems not only by constitutive, synthetic means, but by interpretive ones as well, correlated with experiment, while keeping separate metaphysical and epistemological considerations on the nature of space. In effect, if he gave imagination at large more procedural latitude, he could have granted the possibility of tinkering with revisable mathematical concepts, particularly by exploring the wealth of the pure category of quantity, on quantification and mereological grounds, as well as bracketing our ordinary experience regarding the desirable scope of spatial relations. I hope that this paper, while not exhaustive, indicates the need for further exploration of such a path. To re-interpret the philosophical uses of the doctrine of transcendental idealism in face of the "argument from geometry" is key to this project, in order to defend the investigation on the conceivability of Non-Euclidean objects, in a non-metaphysical realist framework.

II. Constructibility in Historical Context

Janos Bolyai (1802-1860) discovered the possibility of non-Euclidean geometries at the same time that Nikolai Lobachevski (1792-1856) did, in the year of 1823. Yet the great mathematician Carl Gauss (1777-1855) has already anticipated its discovery in 1811. However, by temperament, Gauss was averse to controversy—hence he did not publish it then. He explained why to his friend Bessel the reason in 1829: he was afraid of the philosophers of his time—a time when philosophy was indeed continuous with science, as physics was indeed "*philosophia naturalis*."[5] He was particularly afraid of "metaphysicians" followers of Kant, who had the *Critique of Pure Reason* (1781) as the measure to judge the boundaries of mathematical possibility, spatial and otherwise, with its famous and contentious commitment to Euclidean geometry.

Kant passed away in 1804, thus no real debate with the master was possible afterwards, when the non-Euclidean novelty was finally out in the world. My paper is thus inspired in a dialogue that time-wise could have never been. Nevertheless, *mutatis mutandis*, if such interaction had happened under today's interdisciplinary two-way dialogue between the sciences and philosophy, it could perhaps have taken Kant's work to a completely new direction. This would be the case because I believe Kant was not trying to put forward a dogmatic project, but indeed, a critical one.

Shabel's scholarship contextualizes Kant's work in the history of mathematics.[6] She tells us that the modern mathematics Kant received was the science of magnitude, or quantity, and that the notion of magnitude was undergoing extreme scrutiny with the new methods to study it, such as Descartes' new analytic methods to geometry, Vieta's and Fermat's systems of "specious arithmetic," and Newton and Leibniz's calculus (2005: 30-31). In this ever-changing mathematical landscape, she remarks, philosophers were faced with the task of assessing both their ontological and epistemological tools for explaining the basic notions. The modern use of the term "magnitude" implied a twofold meaning: it described both a quantifiable entity *and* the quantity it was supposed to have. Hence the ontology of modern mathematics included both "abstract mathematical representations and their concrete referents" (ibid.). Accordingly, the questions philosophers inherited from this modern scenario were on how to tackle this dual nature, or, as Shabel puts it, "without giving up to the special status of pure mathematical reasoning, how to explain the ability of pure mathematics to come into contact with and describe the empirically accessible natural world."[7] She distills to a first question on aprioricity—to explain mathematical reasoning on universality, certainty and necessity—and the second question, on its applicability in the natural world. Those were the historical demands Kant had to answer (ibid.).

Dealing with his modern inheritance, Kant responded to (rationalist) mathematicians and metaphysicians alike who were not able to deal with both demands of aprioricity and applicability at the same time, with an integrative approach, in the doctrine of transcendental idealism, which states that space and time are the pure forms of our sensible intuition, and that those are the sources of synthetic a priori cognitions of mathematics proper. Space and time are not "relations of appearances," as Leibniz and Wolff had thought them, but, according to Kant, they are "only creatures of *imagination*, the origin of which must be really sought in experience." (CPR A40/B57)

Shabel wants to emphasize that for Kant the objects of experience ultimately determine our experience of space, and the scope of validity of space and time as forms of sensibility applies only to the objects as appearances (2005: 48). Those eventually are intuited and represented in synthetic a priori cognitions in mathematics. Geometry then, is the paradigmatic case of a science that gives us an a priori representation of space, and as well is applicable to the natural world, conditioned to our scope of validity. It is crucial to notice that for Kant, mathematics provides us with knowledge of the empirical world, not beyond it (Shabel, 2005: 48). Here mathematics seems to relate to our ordinary experience as all there is in the world. But there is a role as well for experiment in revising the mathematical modeling of first order, ordinary experience, which is compatible with a Kantian framework.

III. A Role for Experimental Experience for Pondering About Non-Euclidean Geometries

Despite the constraints given by our ordinary experience we just mentioned, mathematics requires notions that *apparently cannot be found in nature as we experience it*, but that are fundamental for advancing research, such as negative magnitudes, infinitesimals, imaginary roots, and so on. Gauss[8] knew that, and considered these notions to apply beyond ordinary experience[9], in the context of experiment, in order to unleash the most counter-intuitive results compatible with mathematical reasoning, and ultimately to expand the boundaries of our known experiential world (Klein, 1979). Felix Klein says about Gauss that,

> taking Cantor's terminology, for Gauss it was not only a question of the "immanent" but also of the "transient" aspect in mathematics. He was interested not only in the consistent structure of the science for its own sake but also in the possibility of using it to unify and control the

> phenomena of nature. Which ... geometries ... is the most suitable for describing nature, was a question to be decided by experiment. (Klein, 1979: 16)

Gauss worked with applied mathematics, and was unafraid to take on new interpretations of old mathematical concepts to adjust to his work on astronomy and geodesics.[10] Yet even the most specialized experiment relies upon the interpretation of some experiential element by the scientists, according to our human cognitive structure. Hence, here I suggest that there is a possible, understated link between the prudent Gauss and the critical Kant. To illustrate this link, I believe we should consider a broader scope of experimental experience, to inform mathematical reasoning, particularly beyond our ordinary (local, approximate) sense of experience pertaining to Euclidean geometry. In this sense, I am in agreement with Poincaré's conventionalist point of view (Geometry & Experiment, 2001), which is that we shall deny that there is any "property which can...be an absolute criterion enabling us to recognize the straight line, and distinguish it from every other line" (2001: 61), for "in reality our experiments have referred not to space, but to our body and its relations to neighboring objects." (Ibid). It will always be a matter of interpretation and specification pertaining to our situation. To recognize that Euclidean geometry stands in a sort of special relation to us, in our status as navigation solids, helps us to envision ourselves eventually in other "geometrical," such as being two dimensional beings like Flatlanders, and so on.

Kant was aware of the limitation of his powers to relate formalization with our ordinary intuition of space. He knew about the flaws of Euclidean geometry, such as its blatant incompleteness. For, as a deductive system, Euclidean geometry is incomplete in face of the fifth postulate of Euclid, as is common knowledge today. Bolyai, Lobachevski, and Gauss tried to prove the fifth postulate of Euclid by using *reduction ad absurdum* as a method of proof, and thereby they discovered that there were no contradictions in the geometrical systems deriving from its results. Thus, Euclidean geometry was not uniquely possible and had supplementary experimental historical reasons to explore it (Yaglom, 1988: 52-53). Kant in his time, made attempts to prove the fifth parallel postulate as well, and his puzzling conclusions were that it could not be solved mathematically, through construction, but only philosophically, through reasoning about concepts and did not explore it further (Webb, 2006: 280). Kant's conclusion of the impossibility to prove this postulate is a crucial indicator that he was fully aware of the limitations of his mathematical approach and was himself tinkering with alternatives in his system. Webb actually suggests that Kant might have been disingenuous with the difficulties he found carrying out this debate (2006: 231-232). But finally, in his notes published in his *Opus Postumum,* Kant "acknowledges this breach in his system, claiming that mathematics still consists of synthetic a priori propositions but admitting that "if one attempted to progress in this science by proceeding analytically from concepts, one would breach its principles, that is, its formal element as a science within philosophy, although not demonstrating falsely." Apud Webb 2006: 232).

Here in this "breach" opened by the impossibility of proving the fifth postulate of Euclid we may then have some elements for building an argument from Kant to the effect that both discounting locally the role of spatio-temporal predicates and making progress without inconsistency on tinkering with concepts might be an unavoidable supplementary activity in mathematics. In this Kantian "breach" lies the possibility of conceiving non-Euclidean geometries, and separate one or more uses of the term intuition. I suggest that one more degree of epistemic freedom may be granted to the faculty of imagination in Kant's system, for bracketing the role of the intuitive element interpretively in mathematical reasoning.[11] In the case of conceiving non-Euclidean geometries, this freedom would appear by the ability to suspend determinate judgments on our ties to the local, "approximate-to" experience of Euclidean geometry, and by taking any resulting abstracted particulars to have their functions re-interpreted reflexively in the specific context of experimental experience.

IV. A Metaphilosophical Role for Transcendental Idealism and the Pertinence of Intuition

To coordinate these demands for an expanded imagination in a Kantian framework, an important step for us is to consider the fitness of Kant's doctrine of transcendental idealism,[12] so that this project can hold its ground.[13] Despite the fact that this doctrine indeed supports Kant's integrated reply to the modern challenge of mathematics without adhering to a transcendental metaphysics position of his predecessors, there is one more layer of complication: this doctrine is nevertheless said to implicate–in standard interpretation–Euclidean Geometry to be true of his system, as a metaphysical anchor, given its relations with the "the argument from geometry." Such an argument is thus in the center of major interpretative debates among Kant's commentators concerning the robustness of the doctrine of transcendental idealism, and as such it is said to provide a "transcendental" argument in support of the claim that we have a pure intuition of space, in connection to Euclidean geometry, in relation to the transcendental exposition of space.[14] Of course, if Euclidean geometry were for us a fixed and final horizon of mathematical knowledge, it would certainly constitute a serious difficulty for the purpose of understanding the role of geometry in the body of mathematics, including the constraints on this role in mathematical practice, and its origins in intuition. It is hence useful to consider some definite codification of the geometry in question. The idea of Euclidean geometry as being a final representation of our experience and of the world is obviously not acceptable today, and does not agree with the previous contextualist scholarship cited (Shabel, 2004). In this sense, new interpretations of the doctrine of transcendental idealism abandon such construal, by suggesting that either the "argument from geometry" does not have the force to impinge on the fitness of the doctrine itself, as in Allison′s reading,[15] or it may indeed hold, but only in a kind of "synthetic interpretation," as in Shabel′s reading (2004), that is, such an interpretation is not a "regressive" one, which would entail a metaphysical commitment on Kant′s part to Euclidean geometry, but synthetic or progressive.[16] Shabel′s synthetic interpretation broadens and challenges the standard interpretation of the argument by putting together the transcendental and the metaphysical expositions of space[17] to bear weight in the construal of the meaning of the argument. This rendition allows her to show that the proposition that is at stake is restricted to assume that "an a priority concept of space is the source of success of the geometrical method of reasoning" (2004: 205). The point is that, although Euclidean geometry is not the case globally, intuition as codification of Euclidean geometry happens to formalize most of our intuitive spatial relations and Euclidean geometry has for that reason indeed an intuitive source, but in this interpretation, this fact does not mean that Kant takes Euclidean geometry to apply above and beyond our ordinary experience.

Any of these interpretations—Allison′s or Shabel′s—would preserve the doctrine of transcendental idealism itself. The first of these interpretations focuses on making clear the distinction between epistemic and metaphysical readings of Kant′s critical project, in order to keep it properly "critical"—thus, for that matter, to emphasize the doctrine of the ideality of space, in opposition to any metaphysical claim to the existence of Euclidean space as a "thing-in-itself" for Kant.[18] Taking into consideration both interpretations, my overall position on the matter is that Kant′s acceptance of Euclidean geometry, while quite telling of the kind of connection Kant saw between pure intuition and our ordinary experience as the ground of geometry, does not preclude us from updating his system. For this acceptance is not a metaphysical commitment to the applicability of Euclidean geometry to all objects, but instead, shows the degree of editing that our understanding imposes upon our intuitions of space and time as conditions of formalization.[19]

In this sense, transcendental idealism can be interpreted as a metaphilosophical position and, hence, as being on a different level from providing a metaphysical commitment to geometry. As well, in the light of Shabel′s (2005) interpretation is that Kant′s integrative response to the challenge of modern mathematics does not predetermine his or anyone′s definitive view of geometry, given Kant′s starting point in our experience. Far from that, *indeterminate* appearances seem to be the ultimate source of geometrical information abstracted into pure forms of intuition of space and time.[20] Those abstracted features are then taken by Kant to be compatible with the axioms and postulates of Euclidean space (or, for our purposes, as well to non-Euclidean geometries). Here the axiom of intuition as a principle of pure understanding ("All intuitions are extensive magnitudes" [B 202]), helps a Kantian formalize our grasp of magnitude, starting from us experiencing objects in the world, and leading to a discursive formulated geometrical method as it was known then. Thus, the pure intuition of space is indeed a priori, encoded in geometry through its axioms, but nevertheless emerges originally from our experience, so that its products holds true of the world too.

Summing up, Euclidean geometry in this sense is only a kind of formalization of a certain class of spatial intuitions we have, compatible with the local behavior of rigid, solid objects. In this sense, Euclidean geometry cannot be said to pertain in any absolute sense to the world in Kant, but only to a world as intuited in terms of certain spatial relations.

Nevertheless, the doctrine of transcendental idealism is here in fact to preserve the possibility of encoding experience into our intuitive powers, and to keep the critical horizon open, and has nothing to do with the failures in the axiomatization of Euclidean geometry. Thus, its emphasis is more in an epistemological position, rather than a metaphysical or a purely logical one, given Kant's limited means in this area. Accordingly, Allison's *Interpretation and Defense of Kant's Transcendental Idealism* (2004) proposes more broadly that it is in fact an "essentially epistemological or methodological (rather than metaphysical interpretation of...idealism." Moreover, he puts that transcendental idealism relates then to "the discursivity thesis, that is the view that human cognition as discursive requires both concepts and (sensible) intuitions...mark(ing) a radical break with the epistemologies of his predecessors (rationalists and empiricists alike)" (xiv - xv).

In this view, which I endorse, the point Kant is trying to make is a deflationary epistemic one, which takes transcendental idealism at its critical core, not substantive in relation to objects in the world. The main goal of Allison's defense of Kant is to make this distinction very clear, by taking transcendental idealism as a "critical move regarding the scope of spatiotemporal predicates rather than the dogmatic adoption of a subjectivistic metaphysics (2008:1). This position also requires the distinction between theocentric and anthropocentric models of cognition in Kant. Our reflection, he claims, should thus focus on the research on the limits of human cognition, rather than in the metaphysical nature of reality—a defeated theocentric project, that would incur in the foolish attempt to have knowledge beyond the limits of human cognition. Such a metaphysical view would apply to the knowledge of "things-in-themselves." It would be then a "transcendent" as opposed to a "transcendental" approach, in which the domain of the former is unrestricted, while the latter is instead invariably restricted.

Pursuing an epistemic reading, Allison ascribes a special role for intuition arguments as present in the *Transcendental Aesthetics*, in order to consider the appropriate scope of spatiotemporal predicates regarding the so-called "objects in general." Accordingly, he urges that we recognize the restrictive importance of the operation that links representations of space and time to human sensibility, as opposed merely to the understanding. Otherwise, if that operation is not restricted, he suggests that "Kant would have to conclude that they (spatiotemporal predicates) are predicable of things in general," and thus incur in the Wolffian ontological view he wanted to avoid (2008: 27).[2122]

In agreement with Allison's emphasis, I likewise find a place for discussing the epistemic limits, but this time, related to the role of imagination, for devising non-Euclidean geometries according to and restricted by experiment. Hence, this project I defend *is not about allowing for unrestricted operations typical of a theocentric model regarding sensibility to take place, but to re-consider wholesale Kant's ever changing approach to imagination,* so that to permit an extra degree of freedom to both maintain the epistemic open ground in his work, as well as to revise mathematical models—in radical distinction to any metaphysical interpretation thereof.

Notes

I would like to warmly acknowledge the generous feedback and support of my advisor, Prof. Jaakko Hintikka; as well as of Profs. Juliet Floyd and Judson Webb, at Boston University, in the course of my pursuit of this project. I also would like to acknowledge the earlier support of Prof. Iskra Fileva (University of Colorado, Boulder), for her intense philosophical friendship and insight while I researched Kant and schematism, which was my encouragement to later engage on this later project. Naturally, all mistakes are mine, and no one should be liable for my interpretation of Kant. Finally, I would like to thank the financial support of CAPES fellowship for the writing of this article.

1. Notably with Helmholtz's criticism (1977), who argued that Kant's theory of space does not hold in the light of the discovery of the non-Euclidean geometries, as well as by Russell (1991), Reichenbach (1958), and others. The criticism of Kant mounted in such a way is that it is very hard to explore elements of continuity in Kant and non-Euclidean geometries. Parsons (1964: 182) is, to my knowledge, the first to suggest that there is a possible interpretation of the persistence of primitive geometrical elements in Kant's theory of space that would perhaps allow for further exploration of non-Euclidean geometries. I here consider this persistence precisely in imagination.

2. Hilbert has been reported to have remarked about a student who preferred poetry to mathematics: "I never thought he had enough imagination to be a mathematician." In *The Polya Picture*, 1987: 30, apud, Jeremy Kilpatrick, "George Polya's Influence on Mathematics Education" in *Mathematics Magazine* 60.5 (Dec., 1987): 299-300.

3. A complex number can be described as a field, projected from real numbers, through an imaginary unit *i*. This projection allows us to relate a set of real numbers to a complex system, which can be visualized in a diagram, by displaying a real axis and an imaginary axis.

4. The models of hyperbolic geometry generated by Beltrami and Poincare' were also very important for further proving the consistency of hyperbolic geometry itself.

5. See Greenberg, M. (1993: 182).

6. Lisa Shabel (1988, 2004, 2005) received many prizes and distinctions for her historical interpretation of Kant's philosophy of mathematics in the last decades. In this paper she provides much of the historical view of Kant, so that I can keep an exegetic unity in my work.

7. Shabel explains that our contemporary distinction between pure and applied mathematics was not current then, but that it was between pure and mixed mathematics, based on Wolff's conceptualization. Mixed mathematics would include rational thinking on mechanics, astronomy and not least, moral sciences. For further clarification of the history of the concept of mixed mathematics, see Gary Brown, "The History of Mixed Mathematics," *Journal of the History of Ideas* 52.1

On this particular classificatory matter runs "the standard modern complaint against Kant" namely, that Kant did not make the distinction between pure geometry and applied geometry, so that pure geometry could be conceived with no appeal to spatial intuition or other experience, but refer only to axiomatization. Applied geometry, on the other hand, depends upon interpretation in the physical world, settled by empirical evidence. Both Kant's pure and mixed mathematics preserve anachronically the intuitive element in the pure version, which we may consider, if properly encoded, to integrate and resolve unnecessary dualisms. For a matter of lack of space, further elaboration on this point will belong to another paper.

8. Leibniz, before Gauss, would have brought this possibility to life, with what he had called "useful notions," that those should be imagined first, with the help of 'the "natural light of reason" as a stance of pure mathematics, and then checked out later by experience. See Shabel (2005: 48)

9. To further clarify, to bracket ordinary experience seems to be the necessary negative step then to set up the horizon for experiment to weight in, and override habitual expectations on the constitution of our global spatial relations.

10. An example is the case of the concept of "convergent series," Klein narrates, that Gauss was the first one to interpret as if its parts decreased without a limit, and made a distinction with the concept of "convergence" proper, as the partial sums of the series have a limiting value (1979: 50).

11. Here I am committing myself to an epistemological or phenomenological interpretation to the role of intuition in Kant, with Allison, Carlson and others, and taking with a grain of salt Michael Friedman's (*Kant and the exact sciences*, 1992) interpretation that the role of intuition in Kant results from the limitations of the syllogistic logic that Kant had available to work with in his time. By "taking with a grain of salt" I mean that I agree limitations might have certainly played a role in Kant's theoretical choices, but it is indeed not the whole story to be told—there is another path to be observed in Kant's work, which is properly epistemic-phenomenological.

12. "Space is nothing other than merely the form of all appearances of outer sense, i.e., the subjective condition of sensibility, under which alone outer intuition is possible for us.… We can accordingly speak of space, extended beings, and so on, only from the human standpoint. If we depart from the subjective condition under which alone we can acquire outer intuition, namely that through which we may be affected by objects, then the representation of space signifies nothing at all (A27/B43).

13. Among commentators, it has been invariably the case the dismissal of the centrality of the doctrine of transcendental idealism in the context of CPR, since Strawson's influential work (*Bounds of sense*, 1976), where he has put on the table a "separability thesis" in order to make a distinction of Kant's ("good") analytic work from the so-called "defective" metaphysics view of transcendental idealism, in order to allegedly carry thus out with Kant's legacy, *minus* its compromising problems. Here we are actually maintaining the centrality of TI, through Allison's interpretation.

14. *Transcendental Exposition of the Concept of Space*: "Geometry is a science that determines the properties of space synthetically and yet a priori. …Geometrical propositions are all apodictic, i.e., combined with the consciousness of their necessity, e.g., space has only three dimensions; but such propositions cannot be empirical of judgments of experience, nor inferred from them" (CPR B 41).

15. Allison puts, regarding the "Argument from geometry" that "the centrality attributed to geometry by interpreters of Kant's argument for transcendental idealism is misguided. It is granted both that Kant advanced an argument from geometry to the transcendental ideality of space in the transcendental exposition and that his conception of space is intimately connected with his views on geometry. What is denied is merely that Kant's doctrine of the ideality of space is logically dependent on the latter" (2004: 116).

16. See Shabel (2004: 196) "Kant analyzes our synthetic *a priori* knowledge of Euclidean geometry in order to discover what is not yet known: that we have a pure intuition of space. But

this interpretation is in direct conflict with Kant's own stated claim to be providing "synthetic" or "progressive" arguments in the *Critique*, arguments that "develop cognition out of its original seeds without relying on any fact whatever."

Though the ultimate defensibility of Kant's doctrine of transcendental idealism is not my immediate concern in what follows, I propose nevertheless to defend an alternative reading of the "argument from geometry," an argument that I construe as synthetic and so not transcendental in the standard sense. In reinterpreting the "argument from geometry," I thereby reassess the role of geometrical cognition in the arguments of the "Aesthetic," showing that Kant's philosophy of geometry builds a philosophical bridge from his theory of space to his doctrine of transcendental idealism. The "argument from geometry" thereby exemplifies a synthetic argument that reasons progressively *from* a theory of space as pure intuition, offered in the earlier "Metaphysical Exposition of the Concept of Space," to a theory of geometry and, ultimately, to transcendental idealism. Kant's own metaphor will thus guide us in showing that our pure intuition of space provides the seeds for our cognition of the first principles of geometry."

17. (Kant, CRP: B38): "I understand by exposition (expositio) the distinct (even if not complete) representation of that which belongs to a concept; but the exposition is metaphysical when it contains that which exhibits the concept as given a priori."

18. "Our expositions … teach the reality (i.e., objective validity) of space in regard to everything that can come before us as an object, but at the same time, the ideality of space in regard to things when they are considered in themselves through reason, i.e., without taking account of the constitution of our sensibility." (CPR B44/A28)

19. For a more detailed discussion on the role of concept and intuition in Kant, from the perspective we assumed in this paper, see Carlson, Emily (1997: 489-512) who adds to this synthetic view and the misunderstanding on its formalization, by discussing that space cannot be conceptualized and formalized in other ways, but that the grasp Kant is referring to is of *a different order*—intuitive—which cannot be reduced as such to our limitations vis-à-vis the possibilities of monadic logic.

20. The language of indeterminacy used here is a reference to the work of Hopkins, who defends that there is indeterminacy in our relation to space, which might be codified as a kind of visual geometry that would allow for both Euclidean and non-Euclidean intuition to be represented. See Hopkins, James "Visual Geometry," *The Philosophical Review* 82 (1973): 33-34.

21. Allison proposes that Kant actually did an attempt to make clear the distinction of a metaphysical from a cognitive interpretation of transcendental idealism per se from CPR A to B, and by that, to distance himself from a Wolffian interpretation thereof, as he was concerned with a possibly biased reception of the first *Critique*. The latter Wolffian interpretation of the "transcendental" that Kant wanted to avoid, would suggest an ontological reading of transcendental idealism, and in which this biased version ontology per se would be taken to be first philosophy, predicated of objects in general, thus incompatible with the critical spirit of Kant's doctrine. As Allison points out (2008: 4), Kant indeed shows his concerns on this subject in the *Prolegomena*, while in search for a lesser ambiguous language for using in the second edition, with terms like a "formal" or "critical" idealism instead of "transcendental," to distinguish himself from his predecessors (P: 4: 375).