Infinite Analysis and the Connectedness Problem in Leibniz

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Leibniz's rationalism leads him to hold the conceptual containment theory of truth, that is, in any true proposition, the concept of the predicate is contained in or a part of the concept of the subject. Thus all truths for Leibniz are analytic in this sense of analyticity. Of course, the problem is that analytic truths are usually associated with necessary truths, so Leibniz's conceptual containment theory of truth seems to lead to the Spinozistic conclusion that all truths are necessary truths. If the concept of the predicate is in the concept of the subject then it appears that an analysis of the subject will reveal this conceptual connection between subject and predicate concept. Leibniz's famous solution is to say that while in necessary truths the conceptual connection between subject and predicate can be revealed by a finite analysis, in contingent truths, finding the conceptual connection requires an infinite analysis, that is, an analysis that will never end. Thus in *On Freedom* from around 1698 Leibniz writes:

> But in contingent truths, even though the predicate is in the subject, this can never be demonstrated, nor can a proposition ever be reduced to an equality or an identity, but the resolution proceeds to infinity, God alone seeing, not the end of the resolution, of course, which does not exist, but the connection of the terms or the containment of the predicate in the subject, since he sees whatever is in the series. (AG 96)

The kinds of analyses Leibniz has in mind are where the concept of the subject and the concept of the predicate are replaced by equivalent sets of concepts until one has reduced the true proposition in question to what Leibniz calls the form of an identity, explicitly revealing the conceptual containment in question. When one can do this in a finite number of steps Leibniz says there is a demonstration of the proposition, and in what follows I will speak of finite and infinite proofs, reserving the term 'demonstration' for finite proofs. This, of course, is all fairly well-trodden ground in Leibnizian metaphysics of truth.

SOME PROOF PROBLEMS

One problem with all of this, dubbed the Lucky Proof by Robert Adams, is that one wonders why, in the analysis of a contingent truth, one couldn't luckily start the analysis off in such a way that one discovered the predicate in the subject in a finite number of steps (34). A more damning problem, originally stated by Patrick Maher and dubbed the Guaranteed Proof by Gonzalo Rodriguez-Pereyra and Paul Lodge, is that if the predicate is actually in the subject, then one will be guaranteed to discover it in a finite number of steps, no matter how "far off" it is in the analysis (Maher 239, Rodriguez-Pereyra and Lodge 222).

To see why the Guaranteed Proof works one can think of an analogy with the natural numbers. If one thinks of a Leibnizian analysis as the setting out of all the component concepts of a subject in a string, one after another, then, like the natural numbers, that string of concepts will be infinite. However, any given natural number in the infinitely long string will be finitely many steps away from the beginning. Similarly, any predicate in the decomposition string will be finitely many steps away from the beginning of the analysis. So, the Guaranteed Proof reasons, any given true predicate will show up after a finite number of steps in the analysis of its subject.

The Lucky Proof and the Guaranteed Proof, or what I will call the Connectedness Problem, together constitute a serious tangle for Leibniz's infinite analysis account of contingency and it would be nice if there was a way around them. Before proposing my own solution, I want to look at some of the attempts that have been made at a work-around.

NON-CONTAINMENT CONTAINMENT THEORIES

Maher gets around connectedness by denying that the predicate is literally in the subject. Thus in the true contingent proposition "Caesar crossed the Rubicon" what is literally in the subject concept is not the predicate "crossed the Rubicon," but instead the predicate "appearing best to cross the Rubicon" (241). According to Maher, the predicate is "contained" in the subject in the sense that crossing the Rubicon can be derived from appearing best to cross the Rubicon along with the Principle of Perfection—that God has set things up in such a way that substances will freely choose what seems best to them (241). Infinite analysis enters the picture when we ask ourselves about the status of the Principle of Perfection. According to Maher, God chose a world that instantiated the Principle of Perfection because God chose the best possible world, but the predicate, “chooses the best” is not found within the concept of God. Instead, an infinite series of predicates of the form “it seems best to God to choose the best,” and “it seems best to God that it seems best for him to choose the best,” etc. are contained in God (241).

Another solution, due this time to Cover and Hawthorne, shares a feature of Maher’s solution to connectedness, namely, the denial that the predicate is literally in the subject. According to Cover and Hawthorne, macro-predicates like crossing the Rubicon are not literally in the subject. Instead, what are in the subject concept are an infinite number of micro-inclinations (159). A macro-predicate is “contained” in the subject in the sense that one can deduce from all the micro-inclinations what macro-predicate is true of the subject. Infinite analysis enters the picture when one considers that in deducing a given macro-predicate, one has to take account of the infinite number of micro-inclinations that tend toward and against the macro predicate in question, a task that clearly requires an infinite analysis (159).

If the idea behind non-containment theories is to get around connectedness then I think there is little theoretical motivation for them. To see why, consider some true contingent predicate P. The non-containment theorist will say that you will not find P in a finite number of steps because the true contingent predicate will show up nowhere in the analysis of the basic concept, even though the analysis will provide the resources for deducing it.[1] The opponent of the non-containment theorist can reply that one can just take the deductive closure of the basic concept and call that the CIC in some extended sense. Each deduction will then show up as a step of the analysis of this fuller concept and we are faced with the Connectedness Problem all over again. It seems to me that the only thing for the non-containment theorist to do at this point is to deny that deductions should count as steps of an analysis, a move that strikes me as *ad hoc*.

The real work being done to get around connectedness in non-containment theories is not done by the non-containment of the predicate in the subject, but instead by infinitistic considerations. Maher does it by employing an infinite series of reasons for reasons within God for choosing the best, while Cover and Hawthorne do it by having an infinite premise set of micro-inclinations required in order to do any deductive work. While I think that each of these accounts may work, given the lack of textual evidence and the fact that non-containment is not doing any of the real lifting here, I think there is reason to see if we can get around connectedness within the theoretical context of full-blown containment theories.

INFINITE COMPLEXITY OF CIC

Another very interesting attempt around these difficulties, and in the context of full-blown containment, is due to Rodriguez-Pereyra and Lodge. Their basic idea is that connectedness presents no problem because even if one can deduce some true contingent proposition in a finite number of steps, one hasn't proved S is P unless one simultaneously has a proof that the subject concept S is consistent, something that would require a full decomposition of S and therefore could not be accomplished in a finite number of steps (223). This possible solution was considered by Maher originally and again by Cover and Hawthorne and rejected because, among other things, it makes all truths about substances contingent (Maher 239, Cover and Hawthorne 155-56). Rodriguez-Pereyra and Lodge accept this non-intuitive solution and argue that even properties like self-identity are contingent for Leibniz (229).

Rodriguez-Pereyra and Lodge cite passages from *Critical Thoughts on the General Part of the Principles of Descartes* and from *Meditations on Knowledge, Truth, and Ideas* that seem to support their view of how infinitistic considerations enter into Leibniz's containment theory of truth and avoid the Connectedness Problem (228-29). In both of these texts Leibniz argues that the standard Ontological Proof of God's existence is flawed because one has not proved the possibility of a perfect, necessarily existing, being. Hence one cannot safely infer the existence of such a being because the concept might be impossible by secretly containing a contradiction.

It is true that Leibniz insists on this critique of the Ontological Proof many times. Yet I don't find in it any support for Rodriguez-Pereyra and Lodge's thesis. Leibniz insists on the distinction between nominal definitions and real definitions. Nominal definitions merely enable one to distinguish one concept from another, while real definitions establish the possibility of a thing. In *Meditations on Knowledge, Truth, and Ideas*, Leibniz claims that real definitions come in two flavors, a priori and a posteriori. A real definition is had a priori when we have a causal definition of the thing, by which we give the means by which it can be produced mechanically, or we fully analyze it into the primitive attributes of God. A real definition is had a posteriori when we have experience that it exists.[2] The reason that Leibniz insists that we need an a priori proof that the concept of a perfect being is possible is, of course, that none of us has an a posteriori real definition of such a being because none of us has had an experience of the existence of such a being. This is relevantly different than the case of a created being like Caesar. The reason we don't need an a priori proof that the concept of Caesar is consistent is because we have an a posteriori real definition of Caesar, presumably based on the historical records of humans that did have an experience of the existence of such a man.

NON-STANDARD ARITHMETIC

I now want to develop a solution to the Connectedness Problem within a full-blown containment context and, furthermore, avoid such non-intuitive results as making all propositions about created substances contingent. I take my inspiration from some thoughts of Adams. Adams, in trying to make sense of how a contingent predicate could be literally in a subject concept without the denial of that predicate creating a contradiction, appeals to the mathematical notion of ω-inconsistency (26-7). A system that proves for each natural number that it has some property F, and yet also proves that there is a number such that it doesn't have F, is consistent but not ω-consistent.[3] Adams explains that in the same way that a system can be consistent and yet not be ω-consistent, we can view the claim that the denial of a contingent predicate does not generate a contradiction as the view that the denial is consistent but not ω-consistent. He concludes that, "Leibniz reserves 'implies a contradiction' to express a proof-theoretical notion rather than the notion of conceptual falsity" (27). I think this line of thought deserves to be developed.

The notion of ω-consistency is tightly connected to non-standard arithmetic.[4] One way of generating a non-standard model of the Peano axioms is to start adding axioms to the effect that some number 'a' is greater than n, starting with 1 and working one's way up. For each n, this extended system of axioms is, of course, satisfied by the standard model of the natural numbers. This process can be infinitely extended, adding an axiom for each standard natural number. Because of compactness, which states that if every finite subset of an infinite set of sentences has a model, the entire infinite set does, we know that there must be some non-standard model that satisfies the infinitely extended Peano axioms. The reason that you can have a distinction between consistency and ω-consistency is that in this non-standard model, you can prove both that each natural number has some property F and that some non-standard natural number 'a' doesn't have F.

I believe this notion of non-standard arithmetic holds the key to solving the Connectedness Problem. First, the non-standard natural numbers have a non-standard ordering, that is, they are of order type other than ω. In less technical terms they have an ordering that goes 1, 2, 3… and then something more.[5] The other thing to note is that this non-standard ordering of something other than ω means that, in terms of graph theory, they are not connected. Because connectedness is the "mechanism" by which the Guaranteed Proof is being generated, it follows that a non-standard ordering of the predicates in a subject concept is a simple way to solve such difficulties within the context of a full-blown containment theory.

The question remains whether a non-standard ordering is a Leibnizian solution to the Connectedness Problem. I think there is evidence that Leibniz certainly thought that concepts had some kind of internal ordering or structure among their constituent concepts. The following is from the *New Essays*:

> someone who said *The triangle and the trilateral are not the same* would be wrong, since if we consider it carefully we find that the three sides and the three angles always go together…However, one can still say in the abstract that *triangularity is not trilaterality*, or that the formal causes of the triangle and of the trilateral are not the same, as the philosophers put it. They are different aspects of one and the same thing. (NE 363)

Here I take it that Leibniz is saying that although the concepts of triangularity and trilaterality are extensionally equivalent and pick out all the same things, they are intensionally, viewed as concepts, distinct. Given that they don't differ in the objects that they pick out, I think it is fair to conclude that they don't differ in their constituent concepts. The best way for accounting for the difference between the two that I know of is to claim that there is a distinction in the internal ordering of those constituent concepts. So I take this to be textual evidence that Leibniz did believe that concepts had an internal ordering.

The next question I want to ask is whether this solution, which introduces the notion of a non-standard ordering, is, not a solution Leibniz proposed, but rather, whether it is an un-Leibnizian solution. I will close with one last piece of text.

Thus if you say that in an unbounded series there exists no last finite number that can be written in, although there can exist an infinite one: I reply, not even this can exist, if there is no last number. The only other thing I would consider replying to this reasoning is that the number of terms is not always the last number of the series. (RA 101)

Leibniz is wickedly close by approximately 200 years here to the decoupling of the concepts of cardinality and ordinality that is essential to transfinite arithmetic and the discoveries of Georg Cantor.

NOTES

1. There is one other kind of non-containment theory lurking in logical space, what I will refer to as limit theories. The limit theorist maintains that true contingent predicates are not in the subject, but denies that any analysis of the basic concept will provide the resources for deducing the predicate. Instead, the limit theorist maintains that the analysis of the basic concept will infinitely approach the predicate as a limit point. What I say above does not apply to this kind of non-containment theory.

2. See AG 26.

3. That is, the system proves, F1, F2, F3… for each natural number n, and also proves ~Fa for some non-standard natural number a.

4. I am deeply indebted to Richard Grandy for what follows. Any mistakes are my own.

5. In more technical terms they have an ordering that runs 1, 2, 3…, …a-1, a, a+1…, …b-1, b, b+1…, where there are an infinite number of non-standard chunks, …x-1, x, x+1…, which are themselves densely ordered.

ABBREVIATIONS

AG *Philosophical Essays*. Eds. and trans. by R. Ariew and D. Garber. Indianapolis: Hackett, 1989.
NE *New Essays on Human Understanding*. Ed. and trans. by P. Remnant and J. Bennett. NY: Cambridge UP, 1996.
RA *The Labyrinth of the Continuum: Writings on the Continuum Problem, 1672-1686*. Trans. and ed. by Richard Arthur. New Haven: Yale UP, 2001.

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